Exercises with Lecture 1 of Topology in Physics (UvA/Mastermath 2018)

February 6, 2018

This is the sheet of exercises corresponding to the material covered in the first lecture of the 6th of February. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

There are two moments to hand in these exercises. Either on Friday the 9th of February (if using option 3,4 this should be done before 5pm) or before the *lecture* (so not the exercise class) on Tuesday the 13th of February. A selection of the exercises that are handed in on Friday the 9th will be discussed at the beginning of the exercise class on Tuesday the 13th and you will receive the graded/commented exercises during the exercise classes on Tuesday the 20th of February. **In short we offer the option of handing in exercises early so that you may get comments when they are more relevant**.

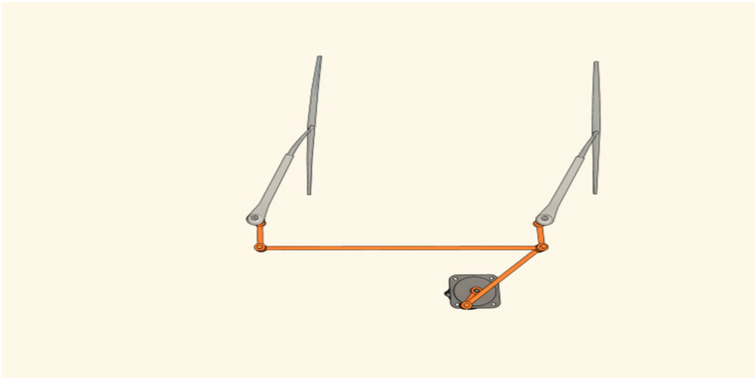

Figure 1: The mechanism behind window wipers [1]

Exercises

Exercise 1

So-called "hinge mechanisms" are mathematically idealized versions of constructions like the windshield wiper fig. 1. They are given by a number of ideal rods and hinges that connect them, some hinges are fixed to the underlying plane while others are only fixing two rods together. The configuration space of a hinge mechanism consists of all the possible configurations such that smooth paths on the configuration space correspond to smooth transitions of configurations. For an illustration consider the animation of the "double pendulum" alongside the 2-torus (its configuration space). Configuration spaces of hinge mechanisms are often smooth manifolds, eg. the n-torus is the configuration space of the n-pendulum. Note in particular that the configurations of an inherently 2-dimensional hinge mechanism can easily give rise to a much higher dimensional manifold.

i Consider the hinge mechanism consisting of 3 rods of length 1 and 1 rod of length $\sqrt{5}$ attached to each other cyclically and such that one of the rods of length 1 attached to the rod of length $\sqrt{5}$ is fixed to the plane at both end points, see figure 2.

Show that the configuration space is isomorphic to S^1. You may show this is a "heuristic" way using computer graphing software if necessary.

ii Consider the same hinge mechanism as in [i], but where all rods are of length 1 (see video).

a Show that the configuration space is a smooth manifold only after removing three points.

[1] image from http://www.etudes.ru/en/etudes/windscreen-wiper/

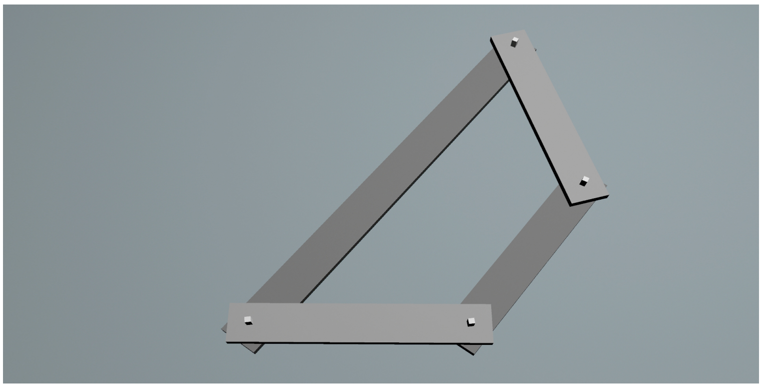

Figure 2: The Hinge mechanism $(1; \sqrt{5}, 1, 1)$.

b Draw the configuration space and the configurations corresponding to the points that should be removed.

c What goes wrong at those points?

Exercise 2

Consider the unit 2-sphere $S^2 \subset \mathbb{R}^3$ given by all points $(x, y, z) \in \mathbb{R}^3$ such that

$$x^2 + y^2 + z^2 = 1$$

and show that it is a smooth manifold by exhibiting an atlas.

(Hint: How does one translate between a globe and a (flat) world map?)

Exercise 3

Suppose M is a smooth manifold equipped with charts $\{U_i\}_{i=1}^r$ such that $M = \cup_{i=1}^r U_i$. Assume the existence of a partition of unity subordinate to $\{U_i\}_{i=1}^r$. Such a partition of unity is given by a collection of smooth functions $\{f_i\}_{i=1}^r$ on M such that $\sum_{i=1}^r f_i(m) = 1$ for all $m \in M$, the functions take values in $[0, 1]$ and $\mathrm{supp} f_i \subset U_i$. The support $\mathrm{supp} f$ of a smooth function f is defined as the closure of the set $\{m \in M | f(m) \neq 0\}$. An embedding $F\colon M \to N$ is a smooth injective map such that $T_x F$ is also injective for all $x \in M$.

i Show that there exists an embedding $\iota\colon M \longrightarrow \mathbb{R}^N$ for $N \in \mathbb{N}$ sufficiently large.

(Hint: It is possible with $N = r + rm$ where m is the dimension of M)

The exercise above asks you to prove an embedding theorem. In fact many more powerful embedding theorems exist. Most notably the Whitney embedding theorem, telling us that we can take $N = 2m$ where m is the dimension of M and the Nash embedding theorem which tells us that we may embed Riemannian manifolds isometrically. Partitions of unity can also be proved to exist in far greater generality and thus the requirement that the manifold M is finitely covered can be removed. The up-shot is that the only manifolds that can occur are submanifolds of $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

ii Explain why it is still useful to consider an intrinsic definition of smooth manifolds that does not refer to some ambient $\mathbb{R}^N$.

$\star$ Exercise 4

The aim of this exercise is to show in an example how the local coordinate (physicists') point of view on differential forms and global coordinate-free (mathematician's) way of thinking can compliment each other. (The moral is: learn both sides!) Let M be a smooth manifold and consider its cotangent bundle T^*M. Construct a 2-form $\omega \in \Omega^2(T^*M)$ on the cotangent bundle in two ways:

i (Local coordinates) Let $(x^1_\alpha, \ldots, x^n_\alpha)$ be local coordinates on $U_\alpha \subset M$. These induce local coordinates $(x^1, \ldots, x^n, p^1, \ldots, p^n)$ on $T^*U_\alpha \subset T^*M$, where the p^i are given as the coefficients of dx^i, i.e. for $\xi \in T^*_xM$ we have $\xi = \sum_{i=1}^n p^i(x)dx^i$. Define

$$\omega|_{T^*U_\alpha} = \sum_i dp_i \wedge dx^i.$$

Show that this transforms correctly under changes in local coordinates to define a global 2-form on T^*M.

ii (Coordinate-free) Let $\pi : T^*M \to M$ be the canonical projection, and use this to define a 1-form λ by

$$\lambda_\theta(X) := \theta(T_\theta\pi(X)), \quad \theta \in T^*M,\ X \in T_\theta(T^*M).$$

Then define $\omega = d\lambda$ show that this defines the same 2-form as in *i*).

Exercise 5

Taken from exercise 4.3.1 in "differential forms in algebraic topology" by R. Bott and L. Tu.

Let $S^n(R) \subset \mathbb{R}^{n+1}$ denote the n-sphere of radius $r \in \mathbb{R}$ defined by the equation

$$(x^1)^2 + \ldots + (x^{n+1})^2 = R^2.$$

Let $\omega_{R,n}$ be the n-form defined by

$$\omega_{R,n} = \sum_{i=1}^{n+1} \frac{(-1)^{i-1}x^i}{R} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1},$$

where the hat signifies omission.

a Compute the integral $\int_{S^n(R)} \omega_{R,n}$ and conclude that $\omega_{R,n}$ is not exact.

b Consider the radius function $r\colon \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow \mathbb{R}$ given by

$$r(x^1, \ldots, x^{n+1}) = \sqrt{(x^1)^2 + \ldots + (x^{n+1})^2}$$

and show that

$$dr \wedge \omega_{r,n} = dx^1 \wedge \ldots \wedge dx^{n+1}.$$

$\star$ Exercise 6

Consider the "double pendulum" hinge mechanism given by two rods of length 1 one of which is attached to the plane at one end point and attached to the other rod at the other end point (see video). Note that a configuration X is specified by the coordinates $(x_1, y_1; x_2, y_2)$ of the end points of the rod that is not fixed to the plane.

i Show that the functions $f_x(X) = x_2 - x_1$ and $f_y(X) = y_2 - y_1$ are smooth.

ii Show that $\alpha := \frac{-1}{f_y} df_x$ is a well-defined closed one-form.

iii Show that α is not exact.

Exercise 7

Definition 1 (Homotopy Equivalence)**.**
A (smooth) homotopy equivalence between two manifolds M and N is given by a pair of smooth maps

$$f\colon M \longrightarrow N \quad \text{and} \quad g\colon N \longrightarrow M$$

such that $f \circ g$ is smoothly homotopic to Id_N and $g \circ f$ is smoothly homotopic to Id_M.

Note that homotopy equivalence defines an equivalence relation on smooth manifolds, which we denote $\sim_h$.

i Show that $N \sim_h M$ implies that $H^\bullet_{dr}(N) \simeq H^\bullet_{dR}(M)$.

ii Complete the proof of the Poincaré lemma.

In this exercise we will compute the cohomology of the 2-torus $\mathbb{T}^2$ by decomposing it into parts of which we already know the cohomology.

iii As preparation we consider the circle $\mathbb{T}^1 := S^1 \hookrightarrow \mathbb{C}$ as embedded into the complex plane. Then let us denote $V_1 := \mathbb{T}^1 - \{1\}$ and $V_i := \mathbb{T}^1 - \{i\}$.

- **a** Show that $V_1 \sim_h \mathbb{R} \sim_h V_i$ and $V_1 \cap V_i \sim_h \mathbb{R} \coprod \mathbb{R}$ (here $\coprod$ denotes the disjoint union).
- **b** Show that $H^\bullet_{dr}(M \coprod N) \simeq H^\bullet_{dR}(M) \oplus H^\bullet_{dR}(N)$.
- **c** Compute the cohomology of V_1, V_i, $V_1 \cap V_i$.
- **d** Use the Mayer–Vietoris sequence to compute the cohomology of $\mathbb{T}^1$.

iv Now let us move on to the 2-torus $\mathbb{T}^2$. We consider the flat model of the 2-torus as the space $\mathbb{R}^2/\mathbb{Z}^2$, i.e. we consider the plane and identify points (x_1, y_1) and (x_2, y_2) if $x_1 - x_2$ and $y_1 - y_2$ are both integers.

- **a** Show that $\mathbb{T}^2$ is given by considering the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ and identifying the points $(0,t)$ with $(1,t)$ for $t \in [0,1]$ as well as identifying the points $(s,0)$ with $(s,1)$ for $s \in [0,1]$.
- **a Bonus** What does this model of $\mathbb{T}^2$ have to do with snake?
- **b** Compute the cohomology of $\mathbb{T}^2$ by decomposing it into two open subsets U_o and U_m such that you already know the cohomology of U_m, U_o and $U_m \cap U_o$ and applying the Mayer–Vietoris sequence.